Chapter 6.1: Drawing graphs in surfaces

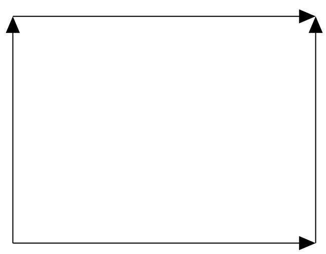
Last time we tried drawing graphs in the plane. First notice it is the same as drawing in a sphere. The same rules apply in terms of edges share only endpoints and no other crossings.

1: Show that a graph is planar if and only if it can be draw on the sphere.

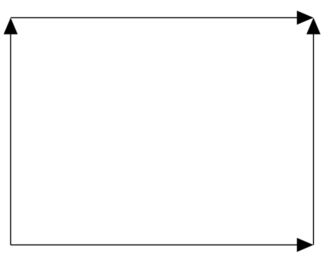
Solution: If you have a graph drawn in the sphere, one can try to unwrap it to the plane - like the globe can be unwrapped into a rectangular map. That is an intuitive explanation. More formally, there is a nice map between the plane and the sphere for all but 1 point. See the textbook.

What about drawing on other surfaces?

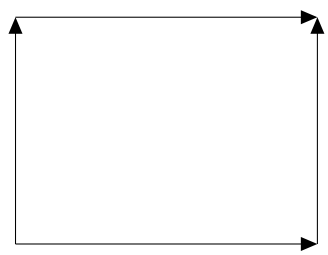
Drawing on the torus is the same as drawing on the following rectangle, where parallel edges with arrows are identified. (This how would you make a torus from a piece of paper if you glue along the opposite edges in the direction of the arrows.)



2: Draw K_5 and $K_{3,3}$ on torus without crossing of edges on torus.



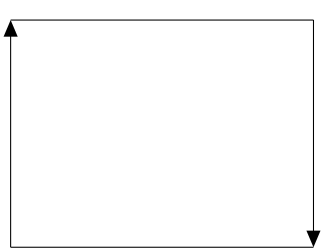
Torus, K_5



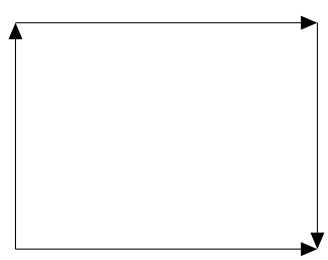
Torus, $K_{3,3}$

Torus can be thought of as adding one handle to a sphere (or adding a bridge to the plane)

Other *famous* surfaces are the Projective plane (Möbius strip) and the Klein bottle.



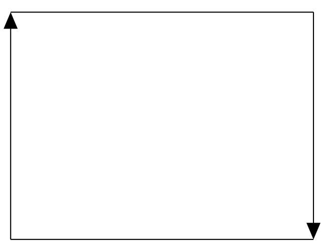
Projective plane



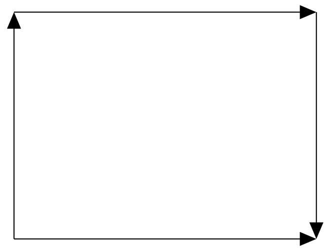
Klein bottle

Projective plane and Klein Bottle are called non-orientable surface since left/right and up/down do not make any sense on these surfaces.

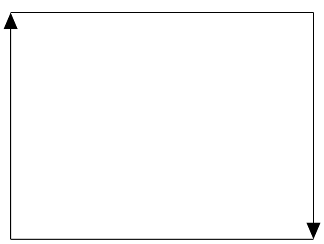
3: Draw graphs in the Projective Plane and in the Klein bottle.



Projective plane, K_5



Klein bottle, K_5



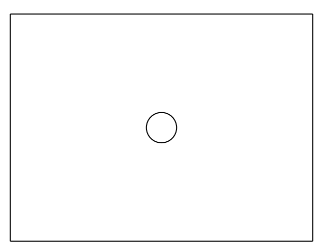
Projective plane, $K_{3,3}$

Klein bottle, $K_{2,2,2}$

An alternative description of the projective plane is to use plane and add a crosscap. A **crosscap** in the plane is point where edges may cross without actually crossing. It can be obtained by cutting a circle in a surface and identifying opposite points.

Also notice that edges going trough a cross cap MUST cross. Arbitrary many edges can cross in one crosscap.

4: Draw K_5 in the plane with one crosscap.



An orientable surface with k handles is usually denoted by S_k.
A non-orientable surface with k cross-caps is usually denoted by N_k.

Let Σ be a surface obtained from the sphere by adding h handles and c cross-caps.

The **Eulerian genus** of Σ is $2h + c$.
Note that the some textbooks defines only orientable surfaces and defines genus of S_k to be k. We use Eulerian genus of S_k, which is $2k$. For surfaces, one handle is equivalent to 2 cross-caps (if there is another cross-cap - Torus is not the same as the Klein bottle)

Classification Theorem for Surfaces
Any closed connected surface is homeomorphic to exactly one of the following surfaces: a sphere, a sphere with finitely many handles, or a sphere with a finitely many crosscaps glued in their place.

Orientable surfaces and Eulerian genus: S_0 plane (0), S_1 torus (2), S_2 double torus (4),...
Non-orientable surfaces and Eulerian genus: N_1 projective plane (1), N_2 Klein bottle (2), ...

Eulerian genus of a graph G is the smallest Eulerian genus of a surface where G can be embedded (without edges crossing).

A region is called a **2-cell** if any closed curve in it can be continuously shrunk to a point (i.e no holes, handles, cross-caps). Embedding of a graph G in a surface is a **2-cell embedding** if every face is 2-cell.

Theorem - Euler for surfaces Let G be a graph embedded in a surface of Euler genus g. Then

$$|V(G)| + |F(G)| \geq |E(G)| + 2 - g,$$

where $F(G)$ is the set of faces of G. With equality if the embedding is a 2-cell embedding.

5: Let G be a graph embedded in a surface of genus g. Find an upper bound on the number of edges of G. (Hint: recall how we did it for planar graphs)

Solution:

$$|V(G)| + |F(G)| \geq |E(G)| + 2 - g$$

Also $2|E| \geq 3|F|$ Hence

$$3|V(G)| + 2|E(G)| \geq 3|E(G)| + 6 - 3g$$

So

$$|E(G)| \leq 3|V(G)| - 6 + 3g$$

Notice that higher genus allows us to add slightly more edges.

Proof can be done by induction on the genus by cutting handles or cross-caps.

Similar as planar graphs, graphs embeddable to a surface of Eulerian genus g can be characterized by a finite set of forbidden minors. There are 35 forbidden minors for projective plane. Thousands for Torus... not all known.

6: Draw K_4 in the projective plane such that every face is a 4-face (bounded by a 4-cycle).

7: Draw Petersen's graph in the projective plane.

8: Draw Petersen's graph on torus.

9: Determine Eulerian genus of K_6

10: Find embedding of K_7 in Torus.

11: What is the largest n such that K_n can be embedded in the projective plane?

12: Find two non-isomorphic embeddings of K_5 on Torus.

13: Prove Euler for surfaces theorem.